Background / Motivation

The following is the product of some playing around I did when TA-ing a topology course. We'd been asked to suggest exam and quiz problems and I was playing around with subspaces of the ordered square I_o (i.e. $[0,1] \times [0,1]$ under the dictionary order topology); they'd just seen it in class and I thought it'd be a rich example to yield exam problems.

I was seeing if deleting some part of the set, looking at the order topology on the resulting subspace and seeing if it leads to anything interesting.

Now the thing about I_o is that it's connected but not path-connected. In a nutshell, path connectivity fails because I_o simply has too many points in it — you can find an uncountable collection of disjoint open intervals, which rules out the possibility of having a continuous path $: [0,1] \to I_o$ from 0×0 to 1×1.

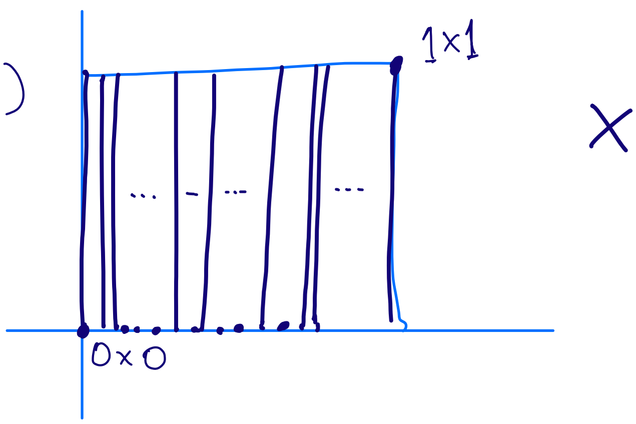
Thus deleting points from I_o becomes a way to counteract this. However you cannot delete too much either, because you might take away enough to disconnect the space. So my goal was to see if I could reach the optimum target — delete enough to ensure path connectedness but not so much as to lose connectedness.

The Actual Problem:

Turns out there is such a space, which turned into my proposed problem: which was that given this space, to show that it is path connected.

The space — let's call it X — is obtained by deleting all the vertical segments $(x \times 0, x \times 1]$ for all irrational x.

So it looks like this:
(everything in dark blue belongs to X)

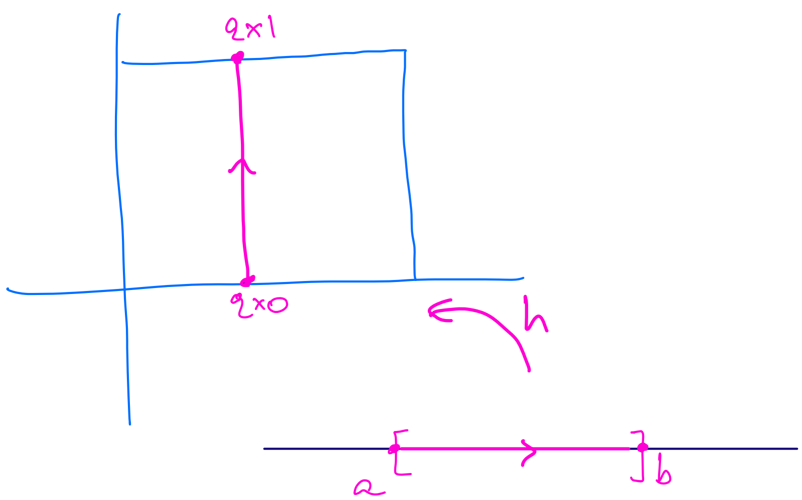


We left the bottommost point $x \times 0$ in so as not to disconnect X.

<u>Notation</u>: For $q \in \mathbb{Q}$, let I_q denote the interval $[q \times 0, q \times 1]$ of X.

We prove path-connectedness of X by constructing a path from 0×0 to 1×1.

First, note that for any interval $[a, b] \subset \mathbb{R}$ and $q \in \mathbb{Q}$ there is a natural map $[a, b] \longrightarrow I_q$ that scales $[a, b]$ by a suitable amount (and maps $a \mapsto q \times 0$, $b \mapsto q \times 1$).

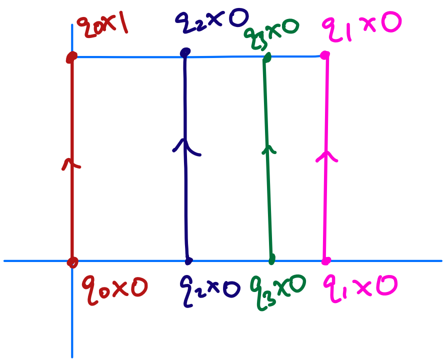


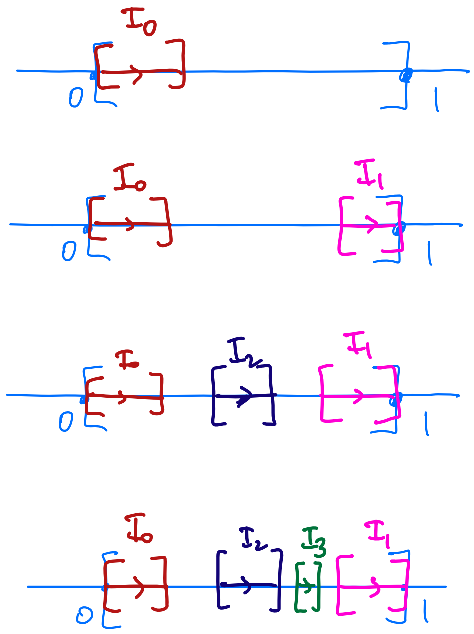
The idea of the construction mimicks the construction in the proof of Urysohn's Lemma (which was done in the course, can be found in Munkres)

Enumerate the rationals in $[0, 1]$ as $q_0, q_1, \ldots$, such that $q_0 = 0$ and $q_1 = 1$.

The idea is to first successively choose a subinterval $[a_i, b_i]$ of $[0, 1]$ which will get mapped to I_{q_i}. Similar to the proof of Urysohn's Lemma, if q_j lies between q_i and q_k then $[a_j, b_j]$ will lie between $[a_i, b_i]$ and $[a_k, b_k]$.

The process for choosing these $[a_i, b_i]$ is illustrated below:





Once this is done, we have to decide where to map the points in what remains of $[0,1]$, i.e. in $[0,1] \setminus \bigcup_{j=1}^{\infty} I_j$.

For such an x, let $l(x) := \sup \{ q_i \mid b_i < x \}$

↳ i.e. q_i s.t. $[a_i, b_i]$ lies to the left of x

and ~~let $r(x) := \inf \{ q_i \mid x < a_i \}$~~

~~↳ i.e. $[a_i, b_i]$ lies to the right of x~~

redundant. Note that we'll always have $l(x) = r(x)$

- If $l(x) \in \mathbb{Q}$, then map x to $l(x) \times 1$ (i.e. the topmost point of the vertical segment corresponding to $l(x)$)

- If $l(x) \notin \mathbb{Q}$, then map x to $l(x) \times 0$

Once this contruction is done, one has to check that the map so obtained is surjective, continuous. Surjectivity is easy; continuity needs to be checked for points in $[0,1] \setminus \bigcup_{j=1}^{\infty} I_j$ and at all a_i and b_i.

This requires some case checking, and is left upto the reader.

Note that $0 \mapsto 0 \times 0$ and $1 \mapsto 1 \times 1$ by this construction, and so we do indeed get a path from 0×0 to 1×1, as desired.

Remark: 1. This problem turned out to be too lengthy to give on an exam, but really all it involves is reusing a familiar example and a familiar proof idea that was seen in the course.

2. One possible problem involving this space X that could perhaps be given on a quiz or exam is to show connectedness (directly, without necessarily showing path connectedness).
Connectedness holds for X for the same reason that it holds for I_o — both are examples of a linear continuum, i.e. a top. space (with an order topology) s.t. the least upper bound property holds, and given $a < b$ $\exists\, c$ s.t. $a < c < b$.
Any such space is always connected.

(See Munkres Ch 3 Section 24)